Classification of States in $O(8)$ Proton-Neutron Pairing Model

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Abstract

Isoscalar ($T = 0$) plus isovector ($T = 1$) pairing hamiltonian in LS-coupling, which is important for heavy N=Z nuclei, is solvable in terms of a $O(8)$ algebra for some special values of the mixing parameter that measures the competition between $T = 0$ and $T = 1$ pairing. The $O(8)$ algebra is generated, amongst others, by the $S = 1, T = 0$ and $S = 0, T = 1$ pair creation and annihilation operators . Shell model algebras, with only number conserving operators, that are complementary to the $O(8) \supset O_{ST}(6) \supset O_S(3) \otimes O_T(3)$, $O(8) \supset [O_S(5) \supset O_S(3)] \otimes O_T(3)$ and $O(8) \supset [O_T(5) \supset O_T(3)] \otimes O_S(3)$ sub-algebras are identified. The problem of classification of states for a given number of

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nucleons (called 'plethysm' problem in group theory), for these group chains is solved explicitly for states with $O(8)$ seniority $v = 0, 1, 2, 3$ and 4. Using them, band structures in isospin space are identified for states with $v = 0, 1, 2$ and 3.

I. INTRODUCTION

In the last few years study of the structure of heavy odd-odd N=Z nuclei (with A $\geq$ 60) near the proton drip line has become an area of intense research as these nuclei are expected to give new insights into neutron-proton (np) correlations (or isospin $T = 0$ vs $T = 1$ pairing) that are hitherto unknown. With the development of radioactive ion beam (RIB) facilities and large detector arrays, there are now experimental results for the energy spectra of ^{62}Ga, ^{66}As, ^{70}Br and ^{74}Rb [1] with many isospin $T = 0$ and $T = 1$ levels identified; in future it is expected that many spectroscopic details of these nuclei will be available. As a result of this, there are several attempts to develop models based on shell model and mean-field methods [2] for understanding and predicting the spectroscopic properties of these and other N=Z odd-odd nuclei in the A=60-100 region. On the other hand models based on group theory, which gave deeper insights into the structure of 'normal' nuclei [3], have also started receiving attention. For example there are also attempts to develop the symmetry schemes of the algebraic interacting boson model with spin(S)-isospin(T) degrees of freedom [4,5] as they will give analytical insights into the structure of these nuclei. The focus in the present paper is on the symmetry schemes of the fully fermionic shell model.

With m-nucleons in shell model orbits $(j_1, j_2, \ldots)$, the spectrum generating algebra (SGA) is $U(2\Omega')$ where $\Omega' = \sum_i (2j_i+1)$ and 2 comes from isospin. A subalgebra that conserves isospin is $U(2\Omega') \supset [U(\Omega') \supset Sp(\Omega')] \otimes SU_T(2)$ and here $Sp(\Omega')$ corresponds to pairing. For identical particles, as it is well

known, this algebra corresponds to quasi-spin $SU(2)$ algebra which generates the seniority quantum number [3]. However for nucleons $Sp(2\Omega')$ corresponds to $O(5)$ and the $O(5)$ algebra is generated only by the isovector pair creation and destruction operators, isospin and the number operators. Hecht and others developed the $O(5)$ algebra in 60's and recently it is being applied to N=Z nuclei [6]. An unsatisfactory aspect of the $O(5)$ model is that it does not contain isoscalar pair operator in its algebra. Therefore it is important to look for an algebra that contains both $T = 0$ and $T = 1$ pair operators. To this end it is necessary to consider shell model in LS-coupling.

Within the nuclear shell model, given the hamiltonian to be a sum of isoscalar and isovector pairing hamiltonians in LS-coupling for the nucleons, it is solvable in some special situations using O(8) algebra and its subalgebras generated amongst others, by the isoscalar ($T = 0$) and isovector ($T = 1$) pair creation and destruction operators. This was first shown by Flowers and Szpikowski in 1964 [7] for the case with nucleons in a single ℓ-shell. The $O(8)$ model admits three group-subgroup chains [8–11]: (i) $O(8) \supset O_{ST}(6) \supset O_S(3) \otimes O_T(3)$; (ii) $O(8) \supset [O_S(5) \supset O_S(3)] \otimes O_T(3)$; (iii) $O(8) \supset [O_T(5) \supset O_T(3)] \otimes O_S(3)$. Section II gives a brief discussion of these three fermionic $O(8)$ symmetry schemes. Classification of the many nucleon states in these three symmetry limits is known only for the $O(8)$ seniority (defined ahead) quantum number $v = 0$ [7–11] (for chain (i) results for $v = 1, 2$ are also known [8,12,11]). Before proceeding further it is important to mention that, as group theory is formidable, recently the Dyson boson mapping is being used [13] to gain new insights into the $O(8)$ model as applied to heavy $N \sim Z$ nuclei.

In LS-coupling, with $\Omega = \sum_i (2\ell_i + 1)$ number of spatial degrees of freedom generated by nucleons in ℓ_1, ℓ_2, ... orbits, the shell model spectrum generating algebra is $U(4\Omega)$ and all its generators are number conserving unlike the $O(8)$ algebra. Therefore it is important to identify the 'complementary' (a notion introduced by Moshinsky and Quesne [14]) number conserving algebras within $U(4\Omega)$, with several ℓ-orbits, corresponding to the three limits of the $O(8)$ model. These algebras, identified in Section III and first reported in [11], are: (A) $U(4\Omega) \supset [U(\Omega) \supset O(\Omega) \supset O_L(3)] \otimes [O_{ST}(6) \supset O_S(3) \otimes O_T(3)]$; (B) $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset \{[O(\Omega) \supset O_L(3)] \otimes SU_T(2)\}] \otimes SU_S(2)$; (C) $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset \{[O(\Omega) \supset O_L(3)] \otimes SU_S(2)\}] \otimes SU_T(2)$. In the reminder of this paper $O_L(3)$ in these group chains is dropped as the results for $O(\Omega) \supset O_L(3)$ sector depend explicitly on the ℓ-orbits appropriate for a given nucleus (see however Section VII). Note that the chains (A), (B) and (C) correspond to the chains (i), (ii) and (iii) respectively of the $O(8)$ model. To the extent that the isoscalar plus isovector pairing is important (i.e. the $O(8)$ limits are fairly good) for heavy $N \sim Z$ nuclei, it is expected that for nuclei with even number of nucleons $O(8)$ states with $v = 0, 2, 4$ and for odd mass nuclei with $v = 1, 3$ will be important. The purpose of this paper is to give complete classification of states in the $O(8)$ model, i.e. for the shell model algebras (A), (B) and (C) for $v \leq 4$ and thus solving a long standing problem. These basis states are associated with the irreducible representations (irreps) of the groups in the three chains. Thus the basic problem is to find the branching rules for irreps of the various group algebras in each chain into irreps of its subalgebras. A powerful way of finding such branching rules is provided by 'plethysms' in group theory. A review of plethysm and its properties can be found in Refs. [15,16]. They are applied

to some of the symmetry schemes of the shell model and the interacting boson model of nuclei in the past [17,5]. However recently there is a new systematic discussion of the plethysm method, development of associated computer codes and some applications to the interacting boson models of atomic nuclei by one of the authors [18,19]. In the present paper, following these recent developments on plethysms [18,19], we have applied this method successfully to the chains (A), (B) and (C) and the results are given in sections IV, V and VI. Finally Section VII gives conclusions.

II. $O(8)$ SYMMETRY SCHEMES

Let us begin with m nucleons in several ℓ orbits ℓ_1, ℓ_2, …. Then the single nucleon states are $a^\dagger_{\ell m_\ell \,;\, \frac{1}{2} m_s \,;\, \frac{1}{2} m_t} \, |0\rangle$ and they are 4Ω in number where $\Omega = \sum_i (2\ell_i + 1)$. For a single ℓ-orbit, pair states are defined by two nucleon states with orbital angular momentum zero ($L = 0$). Then by antisymmetry, two nucleon pair states have spin(S) and isospin (T) to be $(ST) = (10) \oplus (01)$. With this, the isoscalar and isovector pair creation operators $D^\dagger_\mu(\ell)$ and $P^\dagger_\mu(\ell)$ respectively are

$$D^\dagger_\mu(\ell) = \sqrt{\frac{2\ell + 1}{2}} \left(a^\dagger_{\ell \frac{1}{2} \frac{1}{2}} a^\dagger_{\ell \frac{1}{2} \frac{1}{2}} \right)^{0,1,0}_{0,\mu,0} , \quad P^\dagger_\mu(\ell) = \sqrt{\frac{2\ell + 1}{2}} \left(a^\dagger_{\ell \frac{1}{2} \frac{1}{2}} a^\dagger_{\ell \frac{1}{2} \frac{1}{2}} \right)^{0,0,1}_{0,0,\mu} . \qquad (1)$$

Note that we are using (L, S, T) order in (1). For the multi-orbit case one can define the generalized isoscalar and isovector pair operators $D^\dagger_\mu$ and $P^\dagger_\mu$ as linear combinations of single orbit $D^\dagger_\mu(\ell)$'s and $P^\dagger_\mu(\ell)$'s respectively except for phase factors,

$$D^\dagger_\mu = \sum_\ell \beta_\ell \, D^\dagger_\mu(\ell) , \quad P^\dagger_\mu = \sum_\ell \beta_\ell \, P^\dagger_\mu(\ell) ; \quad \beta_\ell = +1 \ \text{ or } \ -1 . \qquad (2)$$

Now it is possible to define the pairing hamiltonian in LS-coupling,

$$H_{pairing}(x) = -(1-x)\sum_\mu P_\mu^\dagger P_\mu \ - \ (1+x)\sum_\mu D_\mu^\dagger D_\mu \ . \tag{3}$$

Note that $P_\mu = (P_\mu^\dagger)^\dagger$ and

$$P_\mu(\ell) = \left(P_\mu^\dagger(\ell)\right)^\dagger = (-1)^\mu \sqrt{(2\ell+1)/2}\,(\tilde{a}_{\ell\frac{1}{2}\frac{1}{2}}\tilde{a}_{\ell\frac{1}{2}\frac{1}{2}})_{0,0,-\mu}^{0,0,1}$$

where $\tilde{a}$ is related to a by

$$a_{\ell m_\ell\,;\,\frac{1}{2}m_s\,;\,\frac{1}{2}m_t} = (-1)^{\ell+1+m_\ell-m_s-m_t}\tilde{a}_{\ell-m_\ell\,;\,\frac{1}{2}-m_s\,;\,\frac{1}{2}-m_t}\ .$$

Similarly D_μ and $D_\mu(\ell)$ are defined. At this stage it is also necessary to define spin (S_μ^1), isospin (T_μ^1), Gamow-Teller $((\sigma\tau)_{\mu,\mu'}^{1,1})$ and number $(\hat{n}$ or the equivalent $Q_0)$ operators,

$$X_{\mu,\mu'}^{S,T} = \sum_\ell \sqrt{2\ell+1}\left(a_{\ell\frac{1}{2}\frac{1}{2}}^\dagger \tilde{a}_{\ell\frac{1}{2}\frac{1}{2}}\right)_{0,\mu,\mu'}^{0,S,T}\ ,$$

$$S_\mu^1 = X_{\mu,0}^{1,0}, \quad T_\mu^1 = X_{0,\mu}^{0,1}, \quad (\sigma\tau)_{\mu,\mu'}^{1,1} = X_{\mu,\mu'}^{1,1}\ , \tag{4}$$

$$\hat{n} = 2\,X_{0,0}^{0,0}, \quad Q_0 = \tfrac{\hat{n}}{2} - \Omega\ .$$

By evaluating the commutators, it can be shown that the 28 operators $P_\mu^\dagger$, P_μ, $D_\mu^\dagger$, D_μ, S_μ^1, T_μ^1, $(\sigma\tau)_{\mu,\mu'}^{1,1}$ and Q_0 generate the following algebras:

$$
\begin{aligned}
O(8) \ &: \ \{P_\mu^\dagger,\ P_\mu,\ D_\mu^\dagger,\ D_\mu,\ S_\mu^1,\ T_\mu^1,\ (\sigma\tau)_{\mu,\mu'}^{1,1},\ Q_0\}\\
O_{ST}(6) \ &: \ \{S_\mu^1,\ T_\mu^1,\ (\sigma\tau)_{\mu,\mu'}^{1,1}\}\\
O_S(5) \ &: \ \{D_\mu^\dagger,\ D_\mu,\ S_\mu^1,\ Q_0\}\\
O_T(5) \ &: \ \{P_\mu^\dagger,\ P_\mu,\ T_\mu^1,\ Q_0\}\\
O_S(3) \ &: \ S_\mu^1\\
O_T(3) \ &: \ T_\mu^1\ .
\end{aligned}
\tag{5}
$$

The $O_{ST}(6)$ algebra in (5) is nothing but Wigner's spin-isospin supermultiplet $SU_{ST}(4)$ algebra [20]. Note that $O_S(3) \sim SU_S(2)$, $O_T(3) \sim SU_T(2)$ and $SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)$. Quadratic Casimir operators of the groups in (5) are (besides S^2 for $O_S(3)$ and T^2 for $O_T(3)$),

$$
\begin{aligned}
C_2(O(8)) &= 2\left(\sum_\mu P_\mu^\dagger P_\mu + \sum_\mu D_\mu^\dagger D_\mu\right) + C_2(O(6)) + Q_0(Q_0 - 6) \, , \\
C_2(O_{ST}(6)) &= S^2 + T^2 + (\sigma\tau)\cdot(\sigma\tau) \, , \\
C_2(O_S(5)) &= 2\sum_\mu D_\mu^\dagger D_\mu + S^2 + Q_0(Q_0 - 3) \, , \\
C_2(O_T(5)) &= 2\sum_\mu P_\mu^\dagger P_\mu + T^2 + Q_0(Q_0 - 3) \, .
\end{aligned}
\tag{6}
$$

In (6), the dot-product is defined by $A^k \cdot B^k = (-1)^k\sqrt{2k+1}(A^k B^k)^0$ and similarly $A^{k_1 k_2} \cdot B^{k_1 k_2}$ is defined. It is seen from (5) that the pairing hamiltonian (3), for any x, has $O(8)$ symmetry as it contains only the generators of $O(8)$. Moreover, by examining the quadratic Casimir invariants in (6), it is seen that the pairing hamiltonian (3) is solvable in the situations $x = 0, 1, -1$ and the corresponding subalgebras (group-subgroup chains) of $O(8)$ are,

$$
\begin{aligned}
x = 0 \ &: \ O(8) \supset O_{ST}(6) \supset O_S(3) \otimes O_T(3) \\
x = 1 \ &: \ O(8) \supset [O_S(5) \supset O_S(3)] \otimes O_T(3) \\
x = -1 \ &: \ O(8) \supset [O_T(5) \supset O_T(3)] \otimes O_S(3)
\end{aligned}
\tag{7}
$$

All these group-subgroup chains have number non-conserving operators.

III. SHELL MODEL ALGEBRAS COMPLEMENTARY TO THE $O(8)$ ALGEBRAS

We will now consider the $O(8)$ chains in (7) in terms of their "complementary" number conserving shell model group chains. In the $(\ell_1, \ell_2, \ldots)^m$ space, there is a $U(4\Omega)$ algebra generated by the operators

$$
u_{\mu_\ell, \mu_S, \mu_T}^{L,S,T}(\ell_1, \ell_2) = \left(a_{\ell_1 \frac{1}{2}\frac{1}{2}}^\dagger \tilde{a}_{\ell_2 \frac{1}{2}\frac{1}{2}}\right)_{\mu_\ell, \mu_S, \mu_T}^{L,S,T} \, .
\tag{8}
$$

With respect to this $U(4\Omega)$, all the m-nucleon states behave as the totally antisymmetric irrep $\{1^m\}$ and this is our starting point.

A. $U(4\Omega) \supset [U(\Omega) \supset O(\Omega)] \otimes [SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)]$ **chain**

With good (LST), $U(4\Omega)$ algebra can be decomposed into a product of space $U(\Omega)$ and spin-isospin $SU_{ST}(4)$ algebras. This gives the group chain,

$$U(4\Omega) \supset [U(\Omega) \supset O(\Omega)] \otimes [SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)] \ . \tag{9}$$

Note that $SU_{ST}(4) \sim O_{ST}(6)$, $SU_S(2) \sim O_S(3)$ and $SU_T(2) \sim O_T(3)$. Following the results in Appendix A and B of [5] it is straight forward to write down the generators and the quadratic Casimir operators (C_2's) of the algebras in (9),

$$U(4\Omega) \ : \ u_{\mu_\ell,\mu_S,\mu_T}^{L,S,T}(\ell_1,\ell_2)$$

$$U(\Omega) \ : \ 2\, u_{\mu_\ell,0,0}^{L,0,0}(\ell_1,\ell_2)$$

$$U_{ST}(4) \ : \ X_{\mu_S,\mu_T}^{S,T} = \sum_\ell \sqrt{2\ell+1}\, u_{0,\mu_S,\mu_T}^{0,S,T}(\ell,\ell)$$

$$O_{ST}(6) \sim SU_{ST}(4) \ : \ X_{\mu_S,\mu_T}^{S,T} \ ; \quad (ST) = (10),(01),(11)$$

$$O(\Omega) \ : \ 2\, u_{\mu_\ell,0,0}^{L=odd,0,0}(\ell,\ell), \ \ V_\mu^L(\ell_1,\ell_2) \ \text{with} \ \ell_1 > \ell_2 \ ;$$

$$\begin{aligned}
&V_\mu^L(\ell_1,\ell_2) = 2\left[\alpha(\ell_1,\ell_2)(-1)^{\ell_1+\ell_2+L}\right]^{\frac{1}{2}} \times \\
&\left\{u_{\mu,0,0}^{L,0,0}(\ell_1,\ell_2) + \alpha(\ell_1,\ell_2)(-1)^L u_{\mu,0,0}^{L,0,0}(\ell_2,\ell_1)\right\} \\
&|\alpha(\ell_1,\ell_2)|^2 = 1, \ \ \alpha(\ell_1,\ell_2)\alpha(\ell_2,\ell_3) = -\alpha(\ell_1,\ell_3)
\end{aligned} \tag{10}$$

$$C_2(U(\Omega)) = 4 \sum_{\ell_1,\ell_2,L} (-1)^{\ell_1+\ell_2} u^{L,0,0}(\ell_1,\ell_2) \cdot u^{L,0,0}(\ell_2,\ell_1) \ ,$$

$$C_2(O(\Omega)) = 8 \sum_{\ell,L=odd} u^{L,0,0}(\ell,\ell) \cdot u^{L,0,0}(\ell,\ell)$$

$$+ \sum_{\ell_1>\ell_2;L} V^L(\ell_1,\ell_2) \cdot V^L(\ell_1,\ell_2) \ ,$$

$$C_2(U_{ST}(4)) = \sum_{S,T} X^{S,T} \cdot X^{S,T} \ .$$

It should be noted that $O(\Omega)$ is not unique and in the multi-orbit case there are several $O(\Omega)$'s as defined by distinct $\alpha(\ell_1, \ell_2)$'s in (10). Using (10), it can be proved that,

$$C_2(U(\Omega)) + C_2(U_{ST}(4)) = \hat{n}(4 + \Omega) ,$$

$$C_2(U_{ST}(4)) = C_2(O_{ST}(6)) + \hat{n}^2/4 , \qquad (11)$$

$$C_2(U(\Omega)) - C_2(O(\Omega)) = 2\left[\sum_\mu P_\mu^\dagger P_\mu + \sum_\mu D_\mu^\dagger D_\mu\right] + \hat{n} .$$

The third equality in (11) is valid only when

$$\beta_{\ell_1}\beta_{\ell_2} = -\alpha(\ell_1, \ell_2) , \quad \ell_1 \neq \ell_2 . \qquad (12)$$

The relations in (12) with β's defining the pair operators in multi-orbit case (see Eq. (2)) and α's defining $O(\Omega)$ generators (see Eq. (10)), via $C_2(O(8))$ given in (6), connect $O(8)$ with $O(\Omega)$. In fact using (6,11) it is seen that,

$$C_2(O(8)) = -C_2(O(\Omega)) + \Omega(\Omega + 6) , \qquad (13)$$

$$\sum_\mu P_\mu^\dagger P_\mu + \sum_\mu D_\mu^\dagger D_\mu = \frac{1}{2}\{-C_2(O(\Omega)) - C_2(O_{ST}(6)) - Q_0(Q_0 - 6) + \Omega(\Omega + 6)\} .$$

$$(14)$$

Thus, clearly the chain (9) is equivalent to the $O(8) \supset O_{ST}(6) \supset O_S(3) \otimes O_T(3)$ chain and it solves the pairing hamiltonian (3) for $x = 0$. One important result that follows from (2,10,12) is that in the multi-orbit case, there are multiple definitions of pair operators P and D as given by (2) and for each of these definitions there is a unique $O(\Omega)$ as defined by (10,12). In all the previous studies involving several orbits, the choice $\beta_\ell = 1$ is made. Also $O(8)$ will not allow for solving the isovector plus isoscalar pairing hamiltonian with β's different for the isoscalar and isovector parts.

In order to construct the spectra generated by the group chain (9), we will now turn to the irreps of the groups in (9) and their reductions. Throughout this article, we use Wybourne's [15] notations $\{--\}$, $[---]$ and $\langle---\rangle$ respectively for denoting $U(N)$, $O(N)$ and $Sp(N)$ irreps.

Our starting point is $\{1^m\}$ irrep of $U(4\Omega)$. Its reduction to irreps of $U(\Omega)$ in (9) is simple as $U(\Omega)$ appears as a factor in the direct product subgroup with other group being $SU_{ST}(4) \sim O_{ST}(6)$ (or $U_{ST}(4)$). The $U_{ST}(4)$ irreps $\{f\} = \{f_1 f_2 f_3 f_4\}$ uniquely define (by transposition) the $U(\Omega)$ irreps [15],

$$U(\Omega) \; : \; \{\tilde{f}\} = \left\{ 4^{f_4} 3^{f_3-f_4} 2^{f_2-f_3} 1^{f_1-f_2} \right\} \; ;$$

$$\sum_i f_i = m, \quad f_1 \geq f_2 \geq f_3 \geq f_4 \geq 0$$

$$O_{ST}(6) \; : \; [P_1, P_2, P_3] =$$

$$\left[\frac{f_1 + f_2 - f_3 - f_4}{2}, \frac{f_1 - f_2 + f_3 - f_4}{2}, \frac{f_1 - f_2 - f_3 + f_4}{2} \right] \; .$$

(15)

Given in (15) are also the equivalent $O_{ST}(6)$ irreps $[P_1, P_2, P_3]$ in terms of the $U_{ST}(4)$ irrep labels. Analogous to the $U(\Omega)$ irreps, one can write the $O(\Omega)$ irreps by introducing the quantum numbers v and $[p_1 p_2 p_3]$ as (see [7]),

$$O(\Omega) \; : \; [\tilde{\mu}] = [4^{\mu_4} 3^{\mu_3-\mu_4} 2^{\mu_2-\mu_3} 1^{\mu_1-\mu_2}] \Leftrightarrow v, [p_1, p_2, p_3] \; ,$$

$$v = \sum_i \mu_i, \quad \mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq 0,$$

$$[p_1, p_2, p_3] =$$

$$\left[\frac{\mu_1 + \mu_2 - \mu_3 - \mu_4}{2}, \frac{\mu_1 - \mu_2 + \mu_3 - \mu_4}{2}, \frac{\mu_1 - \mu_2 - \mu_3 + \mu_4}{2} \right] \; .$$

(16)

With $\{1^m\}_{U(4\Omega)} \rightarrow \{\tilde{f}\}_{U(\Omega)} \otimes [P_1 P_2 P_3]_{O_{ST}(6)}$, the important reduction that is needed is $\{\tilde{f}\}_{U(\Omega)} \rightarrow v[p_1 p_2 p_3]$ of $O(\Omega)$ (of-course $[P_1 P_2 P_3] \rightarrow (ST)$ and $v, [p_1 p_2 p_3] \rightarrow L$ are also needed). Before addressing this problem in Section

IV, let us examine the matrix elements of the quadratic Casimir invariants. Using the general results

$$
\begin{aligned}
\langle C_2(U(N))\rangle^{\{F_1 F_2,\dots\}} &= \sum_i F_i(F_i + N + 1 - 2i) ,\\
\langle C_2(O(N))\rangle^{[\omega_1 \omega_2,\dots]} &= \sum_i \omega_i(\omega_i + N - 2i)
\end{aligned}
\tag{17}
$$

it is seen that

$$
\langle C_2(O(6))\rangle^{[P_1 P_2 P_3]} = P_1(P_1 + 4) + P_2(P_2 + 2) + P_3^2 ,
$$

$$
\langle C_2(O(\Omega))\rangle^{v,[p_1 p_2 p_3]} = v(\Omega + 3 - v/4)
$$
$$
- [p_1(p_1 + 4) + p_2(p_2 + 2) + p_3^2]
\tag{18}
$$

$$
\Rightarrow \langle C_2(O(8))\rangle^{v,[p_1 p_2 p_3]} = Q(Q + 6) + [p_1(p_1 + 4) + p_2(p_2 + 2) + p_3^2] .
$$

where $Q = \Omega - v/2$. The last equality follows from (13). From (14,18) it is clear that the states with $v = 0$ will be lowest in energy for the pairing hamiltonian (3) with $x = 0$. Therefore it is meaningful to consider $v \le 4$ states and workout the allowed $O_{ST}(6)$ irreps. This exercise is carried out in Section IV. Using Eqs. (14) and (18) the spectrum can be constructed. The energies are given by

$$
\langle H_{pairing}(x = 0)\rangle^{m,v,[p],[P],(ST)} = \tfrac{1}{2}\left\{ -\tfrac{1}{4}(m - v)(4\Omega + 12 - m - v) \right.
$$
$$
\left. - [p_1(p_1 + 4) + p_2(p_2 + 2) + p_3^2] + [P_1(P_1 + 4) + P_2(P_2 + 2) + P_3^2] \right\}
\tag{19}
$$

and it is independent of the (ST) quantum numbers.

B. $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset O(\Omega) \otimes SU_T(2)] \otimes SU_S(2)$ **chain**

In order to identify the number conserving group chain that is complementary to $O(8) \supset [O_S(5) \supset O_S(3)] \otimes O_T(3)$, obviously one has to start

with the $U(2\Omega) \otimes SU_S(2)$ subalgebra of $U(4\Omega)$ algebra. As $U(2\Omega)$ contains $O(\Omega) \otimes SU_T(2)$ as a subalgebra, for completing the group chain, one has to find the subalgebras between these two algebras. It is easy to see that $Sp(2\Omega)$ is the subalgebra one is looking for. Therefore the complementary group-subgroup chain is,

$$U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset \{O(\Omega) \otimes SU_T(2)\}] \otimes SU_S(2) . \qquad (20)$$

To establish this result let us consider the generators and the quadratic Casimir operators (C_2's) of the algebras in (20),

$$U(2\Omega) \quad : \quad \sqrt{2} \, u^{L,0,T}_{\mu_\ell,0,\mu_T}(\ell_1,\ell_2)$$

$$U_S(2) \quad : \quad Y^S_{\mu_S} = \sum_\ell \sqrt{2(2\ell+1)} \, u^{0,S,0}_{0,\mu_S,0}(\ell,\ell)$$

$$Sp(2\Omega) \quad : \quad \sqrt{2} \, u^{L,0,T}_{\mu_\ell,0,\mu_T}(\ell,\ell) \, ; \; L+T = odd, \; V^{L,T}_{\mu,\mu_T}(\ell_1,\ell_2) \; \text{with} \; \ell_1 > \ell_2,$$

$$\begin{aligned}
&V^{L,T}_{\mu,\mu_T}(\ell_1,\ell_2) = \sqrt{2}\left[\alpha(\ell_1,\ell_2)(-1)^{\ell_1+\ell_2+L+T}\right]^{\frac{1}{2}} \times \\
&\left\{u^{L,0,T}_{\mu,0,\mu_T}(\ell_1,\ell_2) + \alpha(\ell_1,\ell_2)(-1)^{L+T} u^{L,0,T}_{\mu,0,\mu_T}(\ell_2,\ell_1)\right\} \\
&|\alpha(\ell_1,\ell_2)|^2 = 1, \quad \alpha(\ell_1,\ell_2)\alpha(\ell_2,\ell_3) = -\alpha(\ell_1,\ell_3)
\end{aligned} \qquad (21)$$

$$C_2(U(2\Omega)) = 2 \sum_{\ell_1,\ell_2,L,T} (-1)^{\ell_1+\ell_2} \, u^{L,0,T}(\ell_1,\ell_2) \cdot u^{L,0,T}(\ell_2,\ell_1) ,$$

$$\begin{aligned}
C_2(Sp(2\Omega)) &= 4 \sum_{\ell,L+T=odd} u^{L,0,T}(\ell,\ell) \cdot u^{L,0,T}(\ell,\ell) \\
&\quad + \sum_{\ell_1>\ell_2;L,T} V^{L,T}(\ell_1,\ell_2) \cdot V^{L,T}(\ell_1,\ell_2) ,
\end{aligned}$$

$$C_2(U_S(2)) = \sum_S Y^S \cdot Y^S ,$$

and the $O(\Omega)$ generators and quadratic Casimir operator are given by Eq.(10). It should be noted that again $Sp(2\Omega)$ is not unique in the multi-orbit case and just as $O(\Omega)$, it is defined by distinct $\alpha(\ell_1,\ell_2)$'s in (21). Using (21), it can be proved that,

$$C_2(U(2\Omega)) + C_2(U_S(2)) = \hat{n}(2 + 2\Omega) ,$$

$$C_2(U_S(2)) = 2S^2 + \hat{n}^2/2 , \tag{22}$$

$$C_2(U(2\Omega)) - C_2(Sp(2\Omega)) = 4\left[\sum_\mu D_\mu^\dagger D_\mu\right] - \hat{n} .$$

The third equality in (22) is valid only when (12) is satisfied. Comparing (22) with (6), it is seen that $Sp(2\Omega)$ in (20) is related to $O_S(5)$,

$$C_2(O_S(5)) = -\frac{1}{2}C_2(Sp(2\Omega)) + \Omega(\Omega + 3) , \tag{23}$$

$$2\sum_\mu D_\mu^\dagger D_\mu = -\frac{1}{2}C_2(Sp(2\Omega)) - S^2 - Q_0(Q_0 - 3) + \Omega(\Omega + 3) . \tag{24}$$

Thus, clearly the chain (20) is equivalent to the $O(8) \supset [O_S(5) \supset O_S(3)] \otimes O_T(3)$ chain and it solves the pairing hamiltonian (3) for $x = 1$.

In order to construct the spectra generated by the group chain (20), we will now turn to the irreps of the groups in (20). Just as before, the starting point is $\{1^m\}$ irrep of $U(4\Omega)$. Its reduction to $U(2\Omega)$ irreps is simple as $U(2\Omega)$ in (20) appears in the direct product subgroup with the other group being $SU_S(2)$ (or $U_S(2)$). The $U_S(2)$ irreps $\{f_S\} = \{f_1 f_2\}$ uniquely define (by transposition) the $U(2\Omega)$ irreps,

$$U(2\Omega) : \left\{\tilde{f}_S\right\} = \left\{2^{f_2} 1^{f_1 - f_2}\right\} ; \quad f_1 + f_2 = m, \quad f_1 \geq f_2 \geq 0, \quad S = (f_1 - f_2)/2 . \tag{25}$$

Now the $Sp(2\Omega)$ irreps can be written as,

$$Sp(2\Omega) : \langle\tilde{\mu}_S\rangle = \langle 2^{\mu_1} 1^{\mu_2}\rangle ; \quad v_S = 2\mu_1 + \mu_2 , \quad t_S = \frac{\mu_2}{2} . \tag{26}$$

The v_S and t_S quantum numbers are introduced by examining the eigenvalue expression for $C_2(Sp(2\Omega))$,

$$\langle C_2(Sp(2\Omega))\rangle^{\langle \lambda_1 \lambda_2 \ldots \rangle} = \sum_i \lambda_i (\lambda_i + 2\Omega + 2 - 2i)$$

$$\Rightarrow \quad \langle C_2(Sp(2\Omega))\rangle^{\langle 2^{\mu_1} 1^{\mu_2}\rangle} = \tag{27}$$

$$2\left[\Omega(\Omega + 3) - \left(\Omega - \frac{v_S}{2}\right)\left(\Omega - \frac{v_S}{2} + 3\right) - t_S(t_S + 1)\right] .$$

Using Eqs. (24) and (27) we have,

$$\langle H_{pairing}(x = 1)\rangle^{m,S,v_S,t_S,T,v,[p]} = -\tfrac{1}{4}\,(m - v_S)\,(4\Omega + 6 - m - v_S)$$
$$\tag{28}$$
$$-t_S(t_S + 1) + S(S + 1)$$

and thus the energies in the limit (20) do not depend (explicitly) on T, v and $[p]$ quantum numbers. Eq. (28) shows that for large Ω and fixed S, smallest v_S states will be lowest in energy for the chain (20).

C. $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset O(\Omega) \otimes SU_S(2)] \otimes SU_T(2)$ **chain**

The chain complementary to $O(8) \supset [O_T(5) \supset O_T(3)] \otimes O_S(3)$ is

$$U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset O(\Omega) \otimes SU_S(2)] \otimes SU_T(2) . \tag{29}$$

All results for this chain follow from Section III.B by simply interchanging $S \Leftrightarrow T$. Thus the $U(2\Omega)$ and $Sp(2\Omega)$ algebras in Eq. (29) are defined in orbital-spin space and the $Sp(2\Omega)$ irreps are labeled by (v_T, t_T). The energy formula here is,

$$\langle H_{pairing}(x = -1)\rangle^{m,S,v_T,t_T,T,v,[p]} = -\tfrac{1}{4}\,(m - v_T)\,(4\Omega + 6 - m - v_T)$$
$$\tag{30}$$
$$-t_T(t_T + 1) + T(T + 1) .$$

Therefore chain (29) generates, with the $T(T+1)$ term in Eq. (30), rotational spectra in isospace and thus it is different from chain (20).

IV. IRREP REDUCTIONS FOR THE CHAIN

$$U(4\Omega) \supset [U(\Omega) \supset O(\Omega)] \otimes [SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)]$$

For the $U(4\Omega) \supset [U(\Omega) \supset O(\Omega)] \otimes SU_{ST}(4)$ chain (hereafter called Limit-I), branching rules for $U(\Omega) \supset O(\Omega)$ and $SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)$ (or equivalently $O_{ST}(6) \supset O_S(3) \otimes O_T(3)$) are given in subsections IV.A and IV.B respectively. As discussed in Section III.A, our interest is in obtaining the irreps that belong to $O(8)$ seniority $v \leq 4$.

A. $U(\Omega)$ and $O_{ST}(6)$ irreps for $O(8)$ seniority $v \leq 4$

The branching rules for $U(\Omega) \supset O(\Omega)$ are obtained by finding which 4-column $U(\Omega)$ irreps $\{\tilde{f}\}$ of degree m have a given $O(\Omega)$ irrep $[\tilde{\mu}]$ of degree v in their restriction to $O(\Omega)$. They follow from the rule [15],

$$
\begin{aligned}
\{\tilde{f}\} &= \sum_{\{\tilde{\mu}\}} \sum_{\{\tilde{\delta}\}_{even}} \Gamma_{\{\tilde{\delta}\}\{\tilde{\mu}\}\{\tilde{f}\}} [\tilde{\mu}] \,, \\
\{\tilde{\delta}\} &= \{0\}, \{2\}, \{4\}, \{2^2\}, \{6\}, \{42\}, \{2^3\}, \ldots
\end{aligned}
\tag{31}
$$

In (31) $\Gamma_{\{\tilde{\delta}\}\{\tilde{\mu}\}\{\tilde{f}\}}$ is the multiplicity of $\{\tilde{f}\}$ in the reduction $\{\tilde{\delta}\}\{\tilde{\mu}\} \to \{\tilde{f}\}$. Therefore, the $\{\tilde{f}\}$ irreps we are looking for are those produced by the outer product of the given irrep $\{\tilde{\mu}\}$ by all allowed irreps $\{\tilde{\delta}\}$ with even entries. The irrep $\{\tilde{f}\}$ being a 4-column irrep, so must be $\{\tilde{\delta}\}$ and $\{\tilde{\mu}\}$, then $\{\tilde{\delta}\} = \{4^r, 2^s\}$ with $4r + 2s + v = m$. Note that v is the $O(8)$ seniority quantum number given in Eq. (16). For $v =$ even, one has $m =$ even$= 4k$ and $4k + 2$ where k is a positive integer (similarly ℓ ahead). Then,

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$$m = 4k \Rightarrow s = 2(k - r) - v/2 \Rightarrow$$

$$s = 0, 2, 4, ..., 2k - v/2 \quad \text{for } v = 4\ell$$

$$s = 1, 3, 5, ..., 2k - v/2 \quad \text{for } v = 4\ell + 2$$

$$(32)$$

$$m = 4k + 2 \Rightarrow s = 2(k - r) + 1 - v/2 \Rightarrow$$

$$s = 1, 3, 5, ..., 2k + 1 - v/2 \quad \text{for } v = 4\ell$$

$$s = 0, 2, 4, ..., 2k + 1 - v/2 \quad \text{for } v = 4\ell + 2 .$$

For $v = $ odd, one has $m =$odd$= 4k + 1$ and $4k + 3$. This gives,

$$m = 4k + 1 \Rightarrow s = 2(k - r) - (v - 1)/2 \Rightarrow$$

$$s = 0, 2, 4, ..., 2k - (v - 1)/2 \quad \text{for } v = 4\ell + 1$$

$$s = 1, 3, 5, ..., 2k - (v - 1)/2 \quad \text{for } v = 4\ell + 3$$

$$(33)$$

$$m = 4k + 3 \Rightarrow s = 2(k - r) - (v - 3/2) \Rightarrow$$

$$s = 1, 3, 5, ..., 2k - (v - 3)/2 \quad \text{for } v = 4\ell + 1$$

$$s = 0, 2, 4, ..., 2k - (v - 3)/2 \quad \text{for } v = 4\ell + 3 .$$

Note that $s = 0 \Rightarrow [P] = [p]$. Using the rules of outer products, the results for $v = 0, 1, 2, 3$ and 4 are obtained and they are as follows:

$(i)\ v = 0$

From Eqs.(31) and (32) one has easily,

$$[\tilde{\mu}] = [0] \Rightarrow [\mu] = [0] \Rightarrow [p] = [0] \Rightarrow \left\{\tilde{f}\right\} = \{4^r, 2^s\}$$

$$\Rightarrow \{f\} = \{r + s, r + s, r, r\} \Rightarrow [P] = [s, 0, 0] \ ;$$

$$s = 0, 2, 4, ..., 2k \quad \text{for } m = 4k$$

$$s = 1, 3, 5, ..., 2k + 1 \quad \text{for } m = 4k + 2 .$$

$$(34)$$

$(ii)\ v = 1$

From Eq. (33) one has,

$$s = 0, 2, 4, ..., 2k \quad \text{for } m = 4k + 1$$

$$s = 1, 3, 5, ..., 2k + 1 \quad \text{for } m = 4k + 3 .$$

$(ii.1):$ $[\tilde{\mu}] = [1] \Rightarrow [\mu] = [1, 0, 0, 0] \Rightarrow [p] = [1/2, 1/2, 1/2].$

$$\{\tilde{f}\} = \{4^r, 3, 2^{s-1}\}_{s \geq 1} + \{4^r, 2^s, 1\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s, r+s, r+1, r\} + \{r+s+1, r+s, r, r\} \Rightarrow \tag{35}$$

$$[P] = [s - 1/2, 1/2, -1/2] + [s + 1/2, 1/2, 1/2] .$$

In Eq. (35) the notation $\{\tilde{f}\}_{s \geq s_0}$ means that the irrep $\{\tilde{f}\}$ is present only when $s \geq s_0$ and the same applies to $\{f\}$ and $[P]$ produced by it. This notation is used in the remaining part of this section.

(iii) $v = 2$

From Eq. (32) one has,

$$s = 1, 3, ..., 2k - 1, \quad \text{for } m = 4k$$

$$s = 0, 2, 4, ..., 2k \quad \text{for } m = 4k + 2 .$$

$(iii.1):$ $[\tilde{\mu}] = [2] \Rightarrow [\mu] = [1, 1, 0, 0] \Rightarrow [p] = [1, 0, 0] .$

$$\left\{\tilde{f}\right\} = \{4^{r+1}, 2^{s-1}\}_{s \geq 1} + \{4^r, 3, 2^{s-1}, 1\}_{s \geq 1} + \{4^r, 2^{s+1}\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s, r+s, r+1, r+1\} + \{r+s+1, r+s, r+1, r\} \tag{36}$$

$$+\{r+s+1, r+s+1, r, r\}$$

$$\Rightarrow [P] = [s - 1, 0, 0] + [s, 1, 0] + [s + 1, 0, 0] .$$

$(iii.2):$ $[\tilde{\mu}] = [1^2] \Rightarrow [\mu] = [2, 0, 0, 0] \Rightarrow [p] = [1, 1, 1] .$

$$\left\{\tilde{f}\right\} = \{4^r, 3^2, 2^{s-2}\}_{s \geq 2} + \{4^r, 3, 2^{s-1}, 1\}_{s \geq 1} + \{4^r, 2^s, 1^2\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s, r+s, r+2, r\} + \{r+s+1, r+s, r+1, r\} + \{r+s+2, r+s, r, r\}$$

$$\Rightarrow [P] = [s - 1, 1, -1] + [s, 1, 0] + [s + 1, 1, 1] .$$

$$\tag{37}$$

(iv) $v = 3$

From Eq. (33) one has,

$$s = 1, 3, 5, ..., 2k - 1, \quad \text{for } m = 4k + 1$$

$$s = 0, 2, 4, ..., 2k \quad \text{for } m = 4k + 3 .$$

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$(iv.1):\ [\tilde{\mu}] = [3] \Rightarrow [\mu] = [1,1,1,0] \Rightarrow [p] = [1/2, 1/2, -1/2]$.

$$\{\tilde{f}\} = \{4^{r+1}, 2^{s-1}, 1\}_{s \geq 1} + \{4^r, 3, 2^s\}_{s \geq 0} \Rightarrow$$
$$\{f\} = \{r+s+1, r+s, r+1, r+1\} + \{r+s+1, r+s+1, r+1, r\} \Rightarrow \qquad (38)$$
$$[P] = [s - 1/2, 1/2, 1/2] + [s + 1/2, 1/2, -1/2] \ .$$

$(iv.2):\ [\tilde{\mu}] = [2,1] \Rightarrow [\mu] = [2,1,0,0] \Rightarrow [p] = [3/2, 1/2, 1/2]$.

$$\{\tilde{f}\} = \{4^{r+1}, 3, 2^{s-2}\}_{s \geq 2} + \{4^r, 3^2, 2^{s-2}, 1\}_{s \geq 2} + \{4^{r+1}, 2^{s-1}, 1\}_{s \geq 1}$$
$$+ \{4^r, 3, 2^s\}_{s \geq 1} + \{4^r, 3, 2^{s-1}, 1^2\}_{s \geq 1} + \{4^r, 2^{s+1}, 1\}_{s \geq 0} \Rightarrow$$
$$\{f\} = \{r+s, r+s, r+2, r+1\} + \{r+s+1, r+s, r+2, r\}$$
$$+\{r+s+1, r+s, r+1, r+1\} + \{r+s+1, r+s+1, r+1, r\} \qquad (39)$$
$$+\{r+s+2, r+s, r+1, r\} + \{r+s+2, r+s+1, r, r\} \Rightarrow$$
$$[P] = [s - 3/2, 1/2, -1/2] + [s - 1/2, 3/2, -1/2] + [s - 1/2, 1/2, 1/2]$$
$$+ [s + 1/2, 1/2, -1/2] + [s + 1/2, 3/2, 1/2] + [s + 3/2, 1/2, 1/2] \ .$$

$(iv.3):\ [\tilde{\mu}] = [1^3] \Rightarrow [\mu] = [3,0,0,0] \Rightarrow [p] = [3/2, 3/2, 3/2]$.

$$\{\tilde{f}\} = \{4^r, 3^3, 2^{s-3}\}_{s \geq 3} + \{4^r, 3^2, 2^{s-2}, 1\}_{s \geq 2} + \{4^r, 3, 2^{s-1}, 1^2\}_{s \geq 1} + \{4^r, 2^s, 1^3\}_{s \geq 0} \Rightarrow$$
$$\{f\} = \{r+s, r+s, r+3, r\} + \{r+s+1, r+s, r+2, r\}$$
$$+\{r+s+2, r+s, r+1, r\} + \{r+s+3, r+s, r, r\} \Rightarrow$$
$$[P] = [s - 3/2, 3/2, -3/2] + [s - 1/2, 3/2, -1/2] + [s + 1/2, 3/2, 1/2]$$
$$+ [s + 3/2, 3/2, 3/2] \ .$$

$$(40)$$

$(v)\ v = 4$

From Eq. (32) one has,

$$s = 0, 2, 4, ..., 2k - 2 \quad \text{for } m = 4k$$
$$s = 1, 3, 5, ..., 2k - 1 \quad \text{for } m = 4k + 2 \ .$$

$(v.1):\ [\tilde{\mu}] = [4] \Rightarrow [\mu] = [1,1,1,1] \Rightarrow [p] = [0,0,0]$.

$$\{\tilde{f}\} = \{4^{r+1}, 2^s\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s+1, r+s+1, r+1, r+1\} \Rightarrow \tag{41}$$

$$[P] = [s, 0, 0] .$$

$(v.2): \ [\tilde{\mu}] = [3, 1] \Rightarrow [\mu] = [2, 1, 1, 0] \Rightarrow [p] = [1, 1, 0] .$

$$\{\tilde{f}\} = \{4^{r+1}, 3, 2^{s-2}, 1\}_{s \geq 2} + \{4^{r+1}, 2^s\}_{s \geq 1} + \{4^{r+1}, 2^{s-1}, 1^2\}_{s \geq 1}$$

$$+ \{4^r, 3^2, 2^{s-1}\}_{s \geq 1} + \{4^r, 3, 2^s, 1\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s+1, r+s, r+2, r+1\} + \{r+s+1, r+s+1, r+1, r+1\}$$

$$+ \{r+s+2, r+s, r+1, r+1\} + \{r+s+1, r+s+1, r+2, r\} \tag{42}$$

$$+ \{r+s+2, r+s+1, r+1, r\} \Rightarrow$$

$$[P] = [s-1, 1, 0] + [s, 0, 0] + [s, 1, 1] + [s, 1, -1] + [s+1, 1, 0] .$$

$(v.3): \ [\tilde{\mu}] = [2^2] \Rightarrow [\mu] = [2, 2, 0, 0] \Rightarrow [p] = [2, 0, 0] .$

$$\{\tilde{f}\} = \{4^{r+2}, 2^{s-2}\}_{s \geq 2} + \{4^{r+1}, 3, 2^{s-2}, 1\}_{s \geq 2} + \{4^r, 3^2, 2^{s-2}, 1^2\}_{s \geq 2}$$

$$+ \{4^{r+1}, 2^s\}_{s \geq 1} + \{4^r, 3, 2^s, 1\}_{s \geq 1} + \{4^r, 2^{s+2}\}_{s \geq 0} \Rightarrow$$

$$\{f\} = \{r+s, r+s, r+2, r+2\} + \{r+s+1, r+s, r+2, r+1\}$$

$$+ \{r+s+2, r+s, r+2, r\} + \{r+s+1, r+s+1, r+1, r+1\}$$

$$+ \{r+s+2, r+s+1, r+1, r\} + \{r+s+2, r+s+2, r, r\} \Rightarrow$$

$$[P] = [s-2, 0, 0] + [s-1, 1, 0] + [s, 2, 0] + [s, 0, 0] + [s+1, 1, 0] + [s+2, 0, 0] .$$

$$\tag{43}$$

Note that, for example $[P] = [2, 0, 0]$ appears for $\{\tilde{f}\} = \{4^r, 2^2\}$ and $\{\tilde{f}\} = \{4^{r+1}, 2^2\}$, both for $m = 4k$.

$(v.4): \ [\tilde{\mu}] = [2, 1^2] \Rightarrow [\mu] = [3, 1, 0, 0] \Rightarrow [p] = [2, 1, 1] .$

$$\{\tilde{f}\} = \{4^{r+1}, 3, 2^{s-2}, 1\}_{s\geq 2} + \{4^r, 3^2, 2^{s-1}\}_{s\geq 2} + \{4^r, 3^2, 2^{s-2}, 1^2\}_{s\geq 2} + \{4^{r+1}, 2^{s-1}, 1^2\}_{s\geq 1}$$

$$+ \{4^r, 3, 2^s, 1\}_{s\geq 1} + \{4^r, 3, 2^{s-1}, 1^3\}_{s\geq 1} + \{4^r, 2^{s+1}, 1^2\}_{s\geq 0} \Rightarrow$$

$$\{f\} = \{r+s+1, r+s, r+2, r+1\} + \{r+s+1, r+s+1, r+2, r\}$$

$$+ \{r+s+2, r+s, r+2, r\} + \{r+s+2, r+s, r+1, r+1\}$$

$$+ \{r+s+2, r+s+1, r+1, r\} + \{r+s+3, r+s, r+1, r\}$$

$$+ \{r+s+3, r+s+1, r, r\} \Rightarrow$$

$$[P] = [s-1, 1, 0] + [s, 1, -1] + [s, 2, 0] + [s, 1, 1] + [s+1, 1, 0] + [s+1, 2, 1]$$

$$+ [s+2, 1, 1] \ .$$

$$(44)$$

Note that, for example $[P] = [2, 1, 1]$ appears for $\{\tilde{f}\} = \{4^r, 2, 1^2\}$ and $\{\tilde{f}\} = \{4^{r+1}, 2, 1^2\}$ both for $m = 4k$.

$(v.5): \ [\tilde{\mu}] = [1^4] \Rightarrow [\mu] = [4, 0, 0, 0] \Rightarrow [p] = [2, 2, 2] \ .$

$$\{\tilde{f}\} = \{4^r, 3^4, 2^{s-4}\}_{s\geq 4} + \{4^r, 3^3, 2^{s-3}, 1\}_{s\geq 3} + \{4^r, 3^2, 2^{s-2}, 1^2\}_{s\geq 2}$$

$$+ \{4^r, 3, 2^{s-1}, 1^3\}_{s\geq 1} + \{4^r, 2^s, 1^4\}_{s\geq 0} \Rightarrow$$

$$\{f\} = \{r+s, r+s, r+4, r\} + \{r+s+1, r+s, r+3, r\} + \{r+s+2, r+s, r+2, r\}$$

$$+ \{r+s+3, r+s, r+1, r\} + \{r+s+4, r+s, r, r\} \Rightarrow$$

$$[P] = [s-2, 2, -2] + [s-1, 2, -1] + [s, 2, 0] + [s+1, 2, 1] + [s+2, 2, 2] \ .$$

$$(45)$$

B. $O_{ST}(6) \supset O_S(3) \otimes O_T(3)$ **reductions**

Eqs. (34)-(45) give the allowed $U_{ST}(4)$ or equivalently $O_{ST}(6)$ irreps (i.e. $\{f\}$ or $[P]$) that belong to the $O(8)$ seniority $v \leq 4$. With this, the remaining problem is to obtain the $O_{ST}(6) \supset O_S(3) \otimes O_T(3)$ reductions. To solve this, we consider the equivalent chain $SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)$. Firstly, given

the $U_{ST}(4)$ irrep $\{f\} = \{f_1, f_2, f_3, f_4\}$, the equivalent $SU_{ST}(4)$ irreps is $\{\lambda\} = \{\lambda_1, \lambda_2, \lambda_3\} = \{f_1 - f_4, f_2 - f_4, f_3 - f_4\}$. The general plethysm problem to be solved is $([\frac{1}{2}]'[\frac{1}{2}]'') \otimes \{\lambda\}$. Now, using the simple result that a totally symmetric irrep $\{\sigma\}$ of $U_{ST}(4)$ reduces to the irreps $\{\sigma_1, \sigma_2\}$ and $\{\sigma_1, \sigma_2\}$, with $\sigma_1 + \sigma_2 = \sigma$, of $U_S(2)$ and $U_T(2)$ in $U_{ST}(4) \supset U_S(2) \otimes U_T(2)$ and translating this to $SU_{ST}(4) \supset SU_S(2) \otimes SU_T(2)$, one has

$$\{u\}_{SU(4)} \rightarrow (ST) = \left(\frac{u}{2}, \frac{u}{2}\right) \oplus \left(\frac{u}{2} - 1, \frac{u}{2} - 1\right) \oplus \ldots \oplus \left(\frac{\pi(u)}{2}, \frac{\pi(u)}{2}\right) . \tag{46}$$

In Eq. (46) the symbol $\pi(r)$ is defined by,

$$\pi(r) = 0 \quad \text{if} \quad r = \text{even}, \qquad \pi(r) = 1 \quad \text{if} \quad r = \text{odd} . \tag{47}$$

The reductions for a general $SU_{ST}(4)$ irrep $\{\lambda_1, \lambda_2, \lambda_3\}$ follow by expanding the Schur function $\{\lambda_1, \lambda_2, \lambda_3\}$ in terms of totally symmetric Schur functions and this is symbolically written as [21]

$$\{\lambda_1, \lambda_2, \lambda_3\} = \begin{vmatrix} \{\lambda_1\} & \{\lambda_1 + 1\} & \{\lambda_1 + 2\} \\ \{\lambda_2 - 1\} & \{\lambda_2\} & \{\lambda_2 + 1\} \\ \{\lambda_3 - 2\} & \{\lambda_3 - 1\} & \{\lambda_3\} \end{vmatrix} \tag{48}$$

with $\{0\} = 1$ and $\{-\lambda\} = 0$. Now the basic quantity to compute is $([\frac{1}{2}]'[\frac{1}{2}]'') \otimes \{u_1\}\{u_2\}\{u_3\}$ with $\{u_i\}$'s totally symmetric, and it is nothing but the Kronecker product of $([\frac{1}{2}]'[\frac{1}{2}]'') \otimes \{u_1\}$, $([\frac{1}{2}]'[\frac{1}{2}]'') \otimes \{u_2\}$ and $([\frac{1}{2}]'[\frac{1}{2}]'') \otimes \{u_3\}$. This result together with Eqs. (46) and (48) will completely solve the reduction problem and it is easy to implement this procedure on a machine. It is seen easily from Eqs. (34) to (37) that the $SU(4)$ irreps of interest for $v \le 2$ are of the type $\{s, s\}$, $\{s+1, s\}$, $\{s, s, 1\} \sim \{s, s-1\}$, $\{s+2, s\}$, $\{s, s, 2\} \sim \{s, s-2\}$ and $\{s+1, s, 1\}$. For these irreps, generating the reductions to (ST)'s for $s \le 10$, the following analytic formulas, valid for any u are derived,

$$\{u, u\} \to (ST): \ S + T = u, u - 2, \ldots, \pi(u)$$

$$\{u + 1, u\} \to (ST): \ S + T = u + 1, u, \ldots, 1$$

$$\{u + 2, u\} \to (ST): \begin{cases} (i) \ \ S + T = u, u - 2, \ldots, \pi(u) \\[2ex] (ii) \ \ S + T = r, \ r = u + 2, u + 1, \ldots, 2 \ \begin{cases} S_{max} = r - 1 \\[1ex] T_{max} = r - 1 \end{cases} \end{cases}$$

$$\{u + 1, u, 1\} \to (ST): \begin{cases} (i) \ \ S + T = u, u - 1, \ldots, 1 \\[2ex] (ii) \ \ S + T = r, \ r = u + 1, u, \ldots, 2 \ \begin{cases} S_{max} = r - 1 \\[1ex] T_{max} = r - 1 \end{cases} \end{cases} .$$

$$\tag{49}$$

For Limit-I (i.e. for the chain (9)), as an example, the quantum numbers obtained using Eqs. (34), (36), (37) and (49) for $m = 12$ are shown in Table I for $v = 0$ and Table II for $v = 2$. As seen from Eq. (19), all the (ST) states that belong to a $O(6)$ irrep $[P_1 P_2 P_3]$ are degenerate and the spacing between various $[P_1 P_2 P_3]$ multiplets for $v = 0$ (then $[P_1 P_2 P_3] = [P, 0, 0]$), is $P(P + 4)$ just as in the orbital space in the γ-soft $O(6)$ limit of the interacting boson model (IBM) [22].

V. IRREP REDUCTIONS FOR THE CHAIN
$$U(4\Omega) \supset [U(2\Omega) \supset SP(2\Omega) \supset O(\Omega) \otimes SU_T(2)] \otimes SU_S(2)$$

For the chain $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset O(\Omega) \otimes SU_T(2)] \otimes SU_S(2)$ (hereafter called Limit-II) the branching rules are obtained by using in much more detail the plethysm method. Firstly, the reduction $U(2\Omega) \to Sp(2\Omega)$ is given by Eq.(80) of [15],

$$\{\tilde{f}_S\}_{U(2\Omega)} = \sum_{\{\tilde{\mu}_S\}} \sum_{\{\tilde{\beta}\}} \Gamma_{\{\tilde{\beta}\}\{\tilde{\mu}_S\}\{\tilde{f}_S\}} \langle \tilde{\mu}_S \rangle_{Sp(2\Omega)} \;, \tag{50}$$

$$\{\tilde{\beta}\} = \{0\}, \{1^2\}, \{1^4\}, \{2^2\}, \{1^6\}, \{2^2 1^2\}, \{3^2\}, \dots$$

Note that the $\{\tilde{\beta}\}$ in (50) are the transpose of the $\{\tilde{\delta}\}$ in the reduction $U(n) \rightarrow O(n)$ given by Eq. (31). As pointed out in Section III.B, $\{\tilde{f}_S\}$ and $\langle \tilde{\mu}_S \rangle$ are two columned irreps. Hence $\{\tilde{\beta}\}$ are of the form $\{2^{2\alpha_1} 1^{2\alpha_2}\}$ with α_1 and α_2 being positive integers. Using this simplicity, all the reductions for $U(2\Omega)$ irreps for $m \leq 16$ are generated using Eq. (50) and the results are given in Table III only for $m \leq 12$ so that the table is not too long.

Closely examining the results in Table III, the following band structures are derived for $m \rightarrow S$ and $S \rightarrow (v_S, t_S)$ with $t_S = 0, \frac{1}{2}, 1, \frac{3}{2}$, which are valid for any m,

$$m = \textbf{even}$$

$$S = 0, 1, 2, ..., m/2$$

$$t_S = 0 \Rightarrow v_S = m - 2S, m - 2S - 4, ..., 0 \text{ or } 2$$

$$t_S = 1 \Rightarrow v_S = \begin{cases} (m - 2S + 2, m - 2S, ..., 2)(1 - \delta_{S,0}) \\ m - 2S - 2, m - 2S - 6, ..., 2 \text{ or } 4 \end{cases}$$

$$\tag{51}$$

$$m = \textbf{odd}$$

$$S = 1/2, 3/2, ..., m/2$$

$$t_S = 1/2 \Rightarrow v_S = m + 1 - 2S, m - 1 - 2S, ..., 1$$

$$t_S = 3/2 \Rightarrow v_S = \begin{cases} (3, 5, ..., m + 3 - 2S)(1 - \delta_{S,1/2}) \\ 3, 5, 7, ..., m - 1 - 2S \end{cases} .$$

It is seen from Eq. (51) that for large m, there are two v_S bands for $t_S = 1$

and $3/2$. For finding the branching rules for the $Sp(2\Omega) \rightarrow O(\Omega) \otimes SU_T(2)$ reductions, one first notes that, Eq. (128) of [15] can be used (with $2\ell + 1 = \Omega$ in this equation). Then the irrep $\langle 1 \rangle$ of $Sp(2\Omega)$ reduces into $O(\Omega) \otimes SU_T(2)$ irreps according to

$$\langle 1 \rangle = [1]' \, [1/2]'' \ . \tag{52}$$

Then, by the theorem in Eq. (132) of [15], one obtains that the irreps of $O(\Omega) \otimes SU_T(2)$ contained in the irrep $< \tilde{\mu}_S >$ of $Sp(2\Omega)$ are those contained in the plethysm

$$([1]'[1/2]'')\otimes < \tilde{\mu}_S > \ .$$

Now, these irreps should be converted into Schur functions,

$$\begin{aligned}
[1]' &= \{1\}' \, , \\
[1/2]'' &= \{1\}'' \, , \\
\langle \tilde{\mu}_S \rangle &= \{\tilde{\mu}_S\} + \sum_{\{\rho\}} \left[\sum_{\{\alpha\}} (-1)^{r_\alpha/2} \Gamma_{\{\alpha\}\{\rho\}\{\tilde{\mu}_S\}} \right] \{\rho\} \, .
\end{aligned} \tag{53}$$

The last equation is Eq. (74) of [15]. Also in (53), $\{\alpha\}$ are partitions of the form

$$\begin{pmatrix} a \\ a+1 \end{pmatrix}, \begin{pmatrix} a & b \\ a+1 & b+1 \end{pmatrix}, \ldots$$

in Frobenius notation and $r_\alpha = \alpha_1 + \alpha_2 + \ldots$ is the degree of $\{\alpha\}$. Note that these irreps $\{\alpha\}$ are the transposed of irreps $\{\gamma\}$ in Eq.(64) of [15]. Now $([1]'[1/2]'')\otimes < \tilde{\mu}_S >$ can be written as,

$$\begin{aligned}
([1]'[1/2]'') &\otimes \langle \tilde{\mu}_S \rangle = \\
(\{1\}'\{1\}'') &\otimes \left\{ \{\tilde{\mu}_S\} + \sum_{\{\rho\}} \left[\sum_{\{\alpha\}} (-)^{r_\alpha/2} \Gamma_{\{\alpha\}\{\rho\}\{\tilde{\mu}_S\}} \right] \{\rho\} \right\} = \\
(\{1\}'\{1\}'') &\otimes \{\tilde{\mu}_S\} + \sum_{\{\rho\}} \left[\sum_{\{\alpha\}} (-)^{r_\alpha/2} \Gamma_{\{\alpha\}\{\rho\}\{\tilde{\mu}_S\}} \right] (\{1\}'\{1\}'') \otimes \{\rho\} \, .
\end{aligned} \tag{54}$$

The fact that $\{\tilde{\mu}_S\}$ is a two-column irrep implies, that $\{\alpha\}$ and $\{\rho\}$ also have this property. Then we will use the equation

$$\left\{2^a, 1^b\right\} = \{1^{a+b}\}\{1^a\} - \{1^{a+b+1}\}\{1^{a-1}\}; \quad a > 1 \tag{55}$$

to compute the plethysms that appear in Eq. (54). The net plethysm calculation then will be calculations of type

$$(\{1\}'\{1\}'') \otimes \{1^p\} = \sum_{k_1+k_2=p} \left\{2^{k_2}, 1^{k_1-k_2}\right\}' \{k_1, k_2\}'' \quad \text{for} \quad Sp(2\Omega) \supset O(\Omega) \otimes SU_T(2) .$$
$$\tag{56}$$

After computing the plethysms that appear in Eq. (54) using Eqs. (55) and (56), the resulting Schur functions $\{\mu\}'$ are converted into $O(\Omega)$ irreps using Eq.(69) of [15]. Programming this procedure, the reductions are obtained for $v_S = 1 - 16$ with $v \leq 4$. These results for $v \leq 3$ are given in Tables IV-X. To avoid making the paper too long, the tables for $v = 4$ (i.e. for $[\tilde{\mu}] = [4], [31], [22], [211]$ and $[1111]$) are not given but they can be had upon request from the authors.

Results in Tables IV-X can be arranged, quite strikingly, into band structures in isospin space and they are:

$$v = 0, \quad [\tilde{\mu}] = [0] \rightarrow < \tilde{\mu}_S > = < 2^r >: T = \pi(r), \pi(r) + 2, ..., r, \quad r \geq 0$$

$$v = 1, \quad [\tilde{\mu}] = [1] \rightarrow < \tilde{\mu}_S > = < 2^r 1 >: T = 1/2, 3/2, ..., r + 1/2, \quad r \geq 0$$

$$v = 2, \quad [\tilde{\mu}] = [2] \rightarrow < \tilde{\mu}_S > = < 2^r >: T = \begin{cases} 1, 2, 3, ..., r - 1, \quad r \geq 2 \\ \pi(r), \pi(r) + 2, ..., r, \quad r \geq 1 \end{cases}$$

$$v = 2, \quad [\tilde{\mu}] = [2] \rightarrow < \tilde{\mu}_S > = < 2^r 1^2 >: T = \pi(r), \pi(r) + 2, ..., r, \quad r \geq 0$$

$$v = 2, \quad [\tilde{\mu}] = [1^2] \rightarrow < \tilde{\mu}_S > = < 2^r >: T = 1 - \pi(r), 3 - \pi(r), ..., r - 1, \quad r \geq 1$$

$$v = 2, \quad [\tilde{\mu}] = [1^2] \rightarrow < \tilde{\mu}_S > = < 2^r 1^2 >: T = \begin{cases} 1, 2, 3, ..., r, \quad r \geq 1 \\ 1 - \pi(r), 3 - \pi(r), ..., r + 1, \quad r \geq 0 \end{cases}$$

$$v = 3, \quad [\tilde{\mu}] = [3] \rightarrow < \tilde{\mu}_S > = < 2^r 1 >: T = 1/2, 3/2, ..., r - 1/2, \quad r \geq 1$$

$$v = 3, \quad [\tilde{\mu}] = [21] \rightarrow < \tilde{\mu}_S > = < 2^r 1 >: T = \begin{cases} 1/2, 3/2, ..., r + 1/2, \quad r \geq 1 \\ 1/2, 3/2, ..., r - 1/2, \quad r \geq 2 \\ 3/2, 5/2, ..., r - 3/2, \quad r \geq 3 \end{cases}$$

$$v = 3, \quad [\tilde{\mu}] = [21] \rightarrow < \tilde{\mu}_S > = < 2^r 1^3 >: T = 1/2, 3/2, ..., r + 1/2, \quad r \geq 0$$

$$v = 3, \quad [\tilde{\mu}] = [1^3] \rightarrow < \tilde{\mu}_S > = < 2^r 1 >: T = 1/2, 3/2, ..., r - 1/2, \quad r \geq 1$$

$$v = 3, \quad [\tilde{\mu}] = [1^3] \rightarrow < \tilde{\mu}_S > = < 2^r 1^3 >: T = \begin{cases} 3/2, 5/2, ..., r + 3/2, \quad r \geq 0 \\ 1/2, 3/2, ..., r - 1/2, \quad r \geq 1 \end{cases}$$

$$\tag{57}$$

Combining Eqs. (51) and (57), for fixed v and $[\tilde{\mu}]$ the reductions for $m \rightarrow S$, $S \rightarrow (v_S, t_S)$ and $(v_S, t_S) \rightarrow T$ can be written down. Note that in Eqs. (57), $\langle \tilde{\mu}_S \rangle = \langle 2^{\mu_1} 1^{\mu_2} \rangle \Rightarrow v_S = 2\mu_1 + \mu_2$ and $t_S = \mu_2/2$. Also it seen that for large v_S, the T's form a band.

For Limit-II (i.e. for the chain (20)), as an example, the quantum numbers obtained using Eqs. (51) and (57) for $m = 12$ are shown in Table I for $v = 0$ and Table XI for $v = 2$. Energy formula given by Eq. (28) and the results in Tables I and XI show that (as the energies are independent of isospin T and states with smaller v_S will be lower in energy), the chain (20) generates vibrational type spectra where $v_S/2$ can be viewed as the phonon number. This is similar to the vibrational $U(5)$ limit (in orbital space) of IBM [22]. It is also important to remark that the total set of (ST) values for a given $v, [\tilde{\mu}]$ should be same for all the three limits. This is verified by the results in Table I for $v = 0$. Similarly, this is verified for $v = 2$ (and $[\tilde{\mu}] = [2]$ and $[1^2]$ separately) by the results in Tables II and XI (also by Table XII ahead).

VI. IRREP REDUCTIONS FOR THE CHAIN

$$U(4\Omega) \supset [U(2\Omega) \supset SP(2\Omega) \supset O(\Omega) \otimes SU_S(2)] \otimes SU_T(2)$$

For the chain $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega) \supset O(\Omega) \otimes SU_S(2)] \otimes SU_T(2)$ (hereafter called Limit-III), all the irrep reductions can be obtained by just interchanging S with T in the results in Section V. It is important to stress that in physical applications it is useful to have band structures for a given S. However what we obtain via the results in Section V is $m \to T$, $T \to v_T$ for various allowed t_T, and $(v_T t_T) \to S$ for the $O(\Omega)$ irreps $[\tilde{\mu}] = [0], [1], [2], [1^2], [3], [21], [1^3]$. Using these results, we have constructed tables for $m \to S$, $S \to (v_T, t_T, T)^\alpha$ where α is the multiplicity. Using these tables the following band structures are derived for $v \leq 2$. For $v = 0$,

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$$m = \textbf{even}, \quad [\tilde{\mu}] = [0] \ :$$
$$S = 0, 1, 2, ..., m/2$$
$$S \to t_T = 0, \quad v_T = 2S, 2S+4, ..., m \text{ or } m-2$$
$$v_T \to T = \pi(r), \pi(r)+2, ..., r\,, \quad r = (m - v_T)/2\,. \tag{58}$$

For $v = 1$,

$$m = \textbf{odd}\,, \quad [\tilde{\mu}] = [1] \ :$$
$$S = 1/2, 3/2, ..., m/2$$
$$S \to t_T = 1/2, \quad v_T = 2S, 2S+2, ..., m$$
$$v_T \to T = 1/2, 3/2, \ldots, (m - v_T + 1)/2\,. \tag{59}$$

For $v = 2$,

$$m = \textbf{even}, \quad [\tilde{\mu}] = [2] \ :$$
$$S = 0, 1, 2, ..., m/2$$
$$S \to t_T = 0, \quad v_T = \begin{cases} (2S, 2S+2, \ldots, m)(1 - \delta_{S,0}) \\ 2S+4, 2S+8, \ldots, m \text{ or } m-2 \end{cases}$$
$$v_T \to T = \pi(r), \pi(r)+2, ..., r; \quad r = (m - v_T)/2 \tag{60}$$

$$S \to t_T = 1, \quad v_T = 2S+2, 2S+6, \ldots, m \text{ or } m-2$$
$$v_T \to T = \left\{ \begin{array}{l} 1, 2, 3, ..., r-1\,, \quad r \ge 2 \\ \pi(r), \pi(r)+2, \ldots, r\,, \quad r \ge 0 \end{array} \right\} ; \quad r = (m - v_T + 2)/2\,,$$

$$m = \textbf{even}, \quad [\tilde{\mu}] = [1^2] \ :$$
$$S = 0, 1, 2, ..., m/2;$$
$$S \to t_T = 0, \quad v_T = 2S+2, 2S+6, \ldots, m \text{ or } m-2$$
$$v_T \to T = \pi(r), \pi(r)+2, ..., r; \quad r = (m - v_T)/2$$

$$S \rightarrow t_T = 1, \quad v_T = \begin{cases} (2S, 2S+2, \ldots, m)(1 - \delta_{S,0}) \\ \\ 2S+4, 2S+8, \ldots, m \text{ or } m-2 \end{cases}$$

$$v_T \rightarrow T = \left. \begin{cases} 1, 2, 3, \ldots, r, \quad r \geq 1 \\ \\ 1 - \pi(r), 3 - \pi(r), \ldots, r+1, \quad r \geq 0 \end{cases} \right\} ; \quad r = (m - v_T)/2 . \tag{61}$$

It is also possible to write down the formulas for $v = 3$ but they are not given here.

For Limit-III (i.e. for the chain (29)), as an example, the quantum numbers obtained using Eqs. (58), (60) and (61) for $m = 12$ are shown in Table I for $v = 0$ and Table XII for $v = 2$. Energy formula given by Eq. (30) and the results in Tables I and XII show that, as the energies have $T(T+1)$ dependence and states with smaller v_T will be lower in energy, the chain (29) generates rotational spectra in isospace. This is similar to the rotational $SU(3)$ limit (in orbital space) of IBM [22].

VII. CONCLUSIONS

In this article the long standing problem of classifying low-lying states in the $O(8)$ proton-neutron pairing model is solved. All the results are derived for states with $O(8)$ seniority $v \leq 4$. Extensive use is made of the recent developments [19] in the plethysm method in group theory. Band structures in isospace are also derived in all the three limits of the model for $v \leq 3$. For Limit I (defined by Eq. (9)) they are given by Eqs. (34)-(45) and (49). For Limit II (defined by Eq. (20)) they are given by Eqs. (51) and (57). Finally for Limit III (it is defined by Eq. (29)) they are given by Eqs. (58)-(61). Some typical numerical results are given for $m = 12$ with $v = 0$ and 2 in Tables I, II, XI and XII. From these tables and the energy formulas

given by Eqs. (19), (28) and (30), it is seen that the three symmetry limits, strikingly, generate vibrational, rotational and γ-soft like spectra (as in the three limits of IBM [22]) in isospace.

Throughout this article the $O(\Omega) \supset O_L(3)$ branch is dropped. For completeness we mention that the reductions of the irreps $[\tilde{\mu}]_{O(\Omega)}$, for $v \leq 4$, to L's follow easily by starting with the chain $U(\Omega) \supset O(\Omega) \supset O_L(3)$ and the basic association $\{1\}_{U(\Omega)} \to (\ell_1, \ell_2, \ldots, \ell_r)_{O_L(3)}$ for nucleons occupying the orbits $(\ell_1, \ell_2, \ldots, \ell_r)$; $\Omega = \sum_{i=1}^{r} (2\ell_i + 1)$. Now, Eq. (48) and the known methods [17] for the reduction of symmetric and antisymmetric irreps of $U(\Omega)$ to L's will give the reductions $\{f\}_{U(\Omega)} \to L_{O(3)}$ where $\{f\}$'s are the irreps for $m \leq 4$. These and Eq. (31) will give finally, $[\tilde{\mu}] \to L$ reductions for $v \leq 4$. This procedure is easy to implement on a machine but, as mentioned in Section I, the results here will depend on $(\ell_1, \ell_2, \ldots, \ell_r)$.

With the classification of states available, the next step is to develop Wigner-Racah algebra for the three chains of the $O(8)$ model (for the $O_{ST}(6)$ chain some results are available for $v \leq 2$ [8,12]). With this it is possible to probe the $O(8)$ model in much more detail and apply it to heavy N $\sim$ Z nuclei but this is for future.

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TABLE I. Quantum numbers in the symmetry limits I, II and III [defined by the group chains (9), (20) and (29)] for $m = 12$ and $v = 0$. For the definition of s, see Section IV.

	Limit-I			Limit-II			Limit-III		
s	$[P]_{O(6)}$	$\{f\}_{SU(4)}$	(ST)	S	v_S	T	S	v_T	T
0	[0]	{0}	(00)	0	0	0	0	0	$0,2,4,6$
2	[2]	{22}	(20),(11),(02),		4	$0,2$		4	$0,2,4$
			(00)		8	$0,2,4$		8	$0,2$
4	[4]	{44}	(40),(31),(22),(13),		12	$0,2,4,6$		12	0
			(04),(20),(11),(02),	1	2	1	1	2	$1,3,5$
			(00)		6	$1,3$		6	$1,3$
6	[6]	{66}	(60),(51),(42),(33),		10	$1,3,5$		10	1
			(24),(15),(06),	2	0	0	2	4	$0,2,4$
			(40),(31),(22),(13),		4	$0,2$		8	$0,2$
			(04),(20),(11),(02),		8	$0,2,4$		12	0
			(00)	3	2	1	3	6	$1,3$
					6	$1,3$		10	1
				4	0	0	4	8	$0,2$
					4	$0,2$		12	0
				5	2	1	5	10	1
				6	0	0	6	12	0

TABLE II. Quantum numbers in the symmetry limit-I [defined by the group chain (9)] for $m = 12$ and $v = 2$. For the definition of s see Section IV. In the table $(ST)^r$ indicates that the multiplicity of the irreps (ST) is r.

$[\tilde{\mu}]$	s	$[P]_{O(6)}$	$\{f\}_{SU(4)}$	(ST)
$[2]$	1	$[0]$	$\{0\}$	(00)
		$[11]$	$\{211\}$	$(10), (01), (11)$
		$[2]$	$\{22\}$	$(20), (11), (02), (00)$
	3	$[2]$	$\{22\}$	same as above
		$[31]$	$\{431\}$	$(30), (21)^2, (12)^2, (03), (20), (11)^2, (02), (10), (01),$
				$(31), (22), (13)$
		$[4]$	$\{44\}$	$(40), (31), (22), (13), (04), (20), (11), (02), (00)$
	5	$[4]$	$\{44\}$	same as above
		$[51]$	$\{651\}$	$(50), (41)^2, (32)^2, (23)^2, (14)^2, (05), (40), (31)^2, (22)^2, (13)^2, (04),$
				$(30), (21)^2, (12)^2, (03), (20), (11)^2, (02), (10), (01),$
				$(51), (42), (33), (24), (15)$
		$[6]$	$\{66\}$	$(60), (51), (42), (33), (24), (15), (06),$
				$(40), (31), (22), (13), (04), (20), (11), (02), (00)$
$[1^2]$	1	$[11]$	$\{211\}$	$(10), (01), (11)$
		$[211]$	$\{31\}$	$(10), (01), (21), (12), (11)$
	3	$[21-1]$	$\{332\}$	same as for $\{31\}$
		$[31]$	$\{431\}$	$(30), (21)^2, (12)^2, (03), (20), (11)^2, (02), (10), (01),$
				$(31), (22), (13)$
		$[411]$	$\{53\}$	$(30), (21)^2, (12)^2, (03), (10), (01),$
				$(41), (32), (23), (14), (31), (22), (13), (11)$
	5	$[41-1]$	$\{552\}$	same as for $\{53\}$
		$[51]$	$\{651\}$	$(50), (41)^2, (32)^2, (23)^2, (14)^2, (05), (40), (31)^2, (22)^2, (13)^2, (04),$

$(30), (21)^2, (12)^2, (03), (20), (11)^2, (02), (10), (01),$

$(51), (42), (33), (24), (15)$

[611]   {75} $(61), (52), (43), (34), (25), (16), (51), (42), (33), (24), (15)$

$(50), (41)^2, (32)^2, (23)^2, (14)^2, (05),$

$(31), (22), (13), (30), (21)^2, (12)^2, (03), (11), (10), (01)$

TABLE III. $U(2\Omega) \supset Sp(2\Omega)$ reductions for two columned $U(2\Omega)$ irreps for $m = 0 - 12$.

$\{0\}$	$=$	$< 0 >$
$\{1\}$	$=$	$< 1 >$
$\{2\}$	$=$	$< 2 >$
$\{1^2\}$	$=$	$< 1^2 > + < 0 >$
$\{21\}$	$=$	$< 21 > + < 1 >$
$\{1^3\}$	$=$	$< 1^3 > + < 1 >$
$\{2^2\}$	$=$	$< 2^2 > + < 1^2 > + < 0 >$
$\{21^2\}$	$=$	$< 21^2 > + < 2 > + < 1^2 >$
$\{1^4\}$	$=$	$< 1^4 > + < 1^2 > + < 0 >$
$\{2^21\}$	$=$	$< 2^21 > + < 21 > + < 1^3 > + < 1 >$
$\{21^3\}$	$=$	$< 21^3 > + < 21 > + < 1^3 > + < 1 >$
$\{1^5\}$	$=$	$< 1^5 > + < 1^3 > + < 1 >$
$\{2^3\}$	$=$	$< 2^3 > + < 21^2 > + < 2 >$
$\{2^21^2\}$	$=$	$< 2^21^2 > + < 2^2 > + < 21^2 > + < 1^4 > +2 < 1^2 > + < 0 >$
$\{21^4\}$	$=$	$< 21^4 > + < 21^2 > + < 1^4 > + < 2 > + < 1^2 >$
$\{1^6\}$	$=$	$< 1^6 > + < 1^4 > + < 1^2 > + < 0 >$
$\{2^31\}$	$=$	$< 2^31 > + < 2^21 > + < 21^3 > + < 1^3 > + < 21 > + < 1 >$
$\{2^21^3\}$	$=$	$< 2^21^3 > + < 2^21 > + < 21^3 > + < 1^5 > + < 21 > +2 < 1^3 > + < 1 >$
$\{21^5\}$	$=$	$< 21^5 > + < 21^3 > + < 1^5 > + < 21 > + < 1^3 > + < 1 >$
$\{1^7\}$	$=$	$< 1^7 > + < 1^5 > + < 1^3 > + < 1 >$

Table III (Cont'd)

$$\{2^4\} = <2^4> + <2^21^2> + <1^4> + <2^2> + <1^2> + <0>$$

$$\{2^31^2\} = <2^31^2> + <2^3> + <2^21^2> + <21^4> +2<21^2> + <1^4>$$
$$+ <2> + <1^2>$$

$$\{2^21^4\} = <2^21^4> + <2^21^2> + <21^4> + <1^6> + <2^2> + <21^2>$$
$$+ 2<1^4> +2<1^2> + <0>$$

$$\{21^6\} = <21^6> + <21^4> + <1^6> + <21^2> + <1^4> + <2> + <1^2>$$

$$\{1^8\} = <1^8> + <1^6> + <1^4> + <1^2> + <0>$$

$$\{2^41\} = <2^41> + <2^31> + <2^21^3> + <21^3> + <1^5> + <2^21>$$
$$+ <21> + <1^3> + <1>$$

$$\{2^31^3\} = <2^31^3> + <2^31> + <2^21^3> + <21^5> + <2^21> +2<21^3>$$
$$+ <1^5> +2<1^3> + <21> + <1>$$

$$\{2^21^5\} = <2^21^5> + <2^21^3> + <21^5> + <1^7> + <2^21> + <21^3>$$
$$+ 2<1^5> + <21> +2<1^3> + <1>$$

$$\{21^7\} = <21^7> + <21^5> + <1^7> + <21^3> + <1^5> + <21> + <1^3>$$
$$+ <1>$$

$$\{1^9\} = <1^9> + <1^7> + <1^5> + <1^3> + <1>$$

$$\{2^5\} = <2^5> + <2^31^2> + <21^4> + <2^3> + <21^2> + <2>$$

$$\{2^41^2\} = <2^41^2> + <2^4> + <2^31^2> + <2^21^4> +2<2^21^2> + <21^4>$$
$$+ <1^6> +2<1^4> + <2^2> + <21^2> +2<1^2> + <0>$$

$$\{2^31^4\} = <2^31^4> + <2^31^2> + <2^21^4> + <21^6> + <2^3> + <2^21^2>$$
$$+ 2<21^4> + <1^6> +2<21^2> +2<1^4> + <2> + <1^2>$$

$$\{2^21^6\} = <2^21^6> + <2^21^4> + <21^6> + <1^8> + <2^21^2> + <21^4>$$
$$+ 2<1^6> + <2^2> + <21^2> +2<1^4> +2<1^2> + <0>$$

Table III (Cont'd)

$\{21^8\} = <21^8> + <21^6> + <1^8> + <21^4> + <1^6> + <21^2> + <1^4>$

$\qquad + <2> + <1^2>$

$\{1^{10}\} = <1^{10}> + <1^8> + <1^6> + <1^4> + <1^2> + <0>$

$\{2^5 1\} = <2^5 1> + <2^4 1> + <2^3 1^3> + <2^2 1^3> + <21^5> + <2^3 1>$

$\qquad + <1^5> + <2^2 1> + <21^3> + <1^3> + <21> + <1>$

$\{2^4 1^3\} = <2^4 1^3> + <2^4 1> + <2^3 1^3> + <2^2 1^5> + <2^3 1> + 2<2^2 1^3>$

$\qquad + <21^5> + <1^7> + 2<21^3> + 2<1^5> + <2^2 1> + <21>$

$\qquad + 2<1^3> + <1>$

$\{2^3 1^5\} = <2^3 1^5> + <2^3 1^3> + <2^2 1^5> + <21^7> + <2^3 1> + <2^2 1^3>$

$\qquad + 2<21^5> + <1^7> + <2^2 1> + 2<21^3> + 2<1^5> + 2<1^3>$

$\qquad + <21> + <1>$

$\{2^2 1^7\} = <2^2 1^7> + <2^2 1^5> + <21^7> + <1^9> + <2^2 1^3> + <21^5>$

$\qquad + 2<1^7> + <2^2 1> + <21^3> + 2<1^5> + <21> + 2<1^3> + <1>$

$\{21^9\} = <21^9> + <21^7> + <1^9> + <21^5> + <1^7> + <21^3>$

$\qquad + <1^5> + <21> + <1^3> + <1>$

$\{1^{11}\} = <1^{11}> + <1^9> + <1^7> + <1^5> + <1^3> + <1>$

$\{2^6\} = <2^6> + <2^4 1^2> + <2^2 1^4> + <2^4> + <1^6> + <2^2 1^2>$

$\qquad + <1^4> + <2^2> + <1^2> + <0>$

$\{2^5 1^2\} = <2^5 1^2> + <2^5> + <2^4 1^2> + <2^3 1^4> + 2<2^3 1^2> + <2^2 1^4>$

$\qquad + <21^6> + 2<21^4> + <1^6> + <2^3> + <2^2 1^2> + 2<21^2> + <1^4>$

$\qquad + <2> + <1^2>$

$\{2^4 1^4\} = <2^4 1^4> + <2^4 1^2> + <2^3 1^4> + <2^2 1^6> + <2^4> + <2^3 1^2>$

$\qquad + 2<2^2 1^4> + <21^6> + <1^8> + 2<2^2 1^2> + 2<21^4> + 2<1^6>$

$\qquad + 3<1^4> + <2^2> + <21^2> + 2<1^2> + <0>$

Table III (Cont'd)

$\{2^3 1^6\} = <2^3 1^6> + <2^3 1^4> + <2^2 1^6> + <21^8> + <2^3 1^2> + <2^2 1^4>$

$\quad + 2<21^6> + <1^8> + <2^3> + <2^2 1^2> +2<21^4> +2<1^6>$

$\quad + 2<21^2> +2<1^4> + <2> + <1^2>$

$\{2^2 1^8\} = <2^2 1^8> + <2^2 1^6> + <21^8> + <1^{10}> + <2^2 1^4> + <21^6>$

$\quad + 2<1^8> + <2^2 1^2> + <21^4> +2<1^6> + <2^2> + <21^2>$

$\quad + 2<1^4> +2<1^2> + <0>$

$\{21^{10}\} = <21^{10}> + <21^8> + <1^{10}> + <21^6> + <1^8> + <21^4> + <1^6>$

$\quad + <21^2> + <1^4> + <2> + <1^2>$

$\{1^{12}\} = <1^{12}> + <1^{10}> + <1^8> + <1^6> + <1^4> + <1^2> + <0>$

TABLE IV. $Sp(2\Omega)$ irreps $< \tilde{\mu}_S >$ that contain the $O(\Omega)$ irrep $[0]$, with multiplicities and associated $(2T+1)$ values given in column 3.

v_S	$< \tilde{\mu}_S >$	$[\tilde{\mu}] = [0]$ $\alpha(2T+1)$
0	$< 0 >$	1(1)
2	$< 2 >$	1(3)
4	$< 2^2 >$	1(5) 1(1)
6	$< 2^3 >$	1(7) 1(3)
8	$< 2^4 >$	1(9) 1(5) 1(1)
10	$< 2^5 >$	1(11) 1(7) 1(3)
12	$< 2^6 >$	1(13) 1(9) 1(5) 1(1)
14	$< 2^7 >$	1(15) 1(11) 1(7) 1(3)
16	$< 2^8 >$	1(17) 1(13) 1(9) 1(5) 1(1)

TABLE V. $Sp(2\Omega)$ irreps $< \tilde{\mu}_S >$ that contain the $O(\Omega)$ irrep $[1]$, with multiplicities and associated $(2T + 1)$ values given in column 3.

v_S	$< \tilde{\mu}_S >$	$[\tilde{\mu}] = [1]$ $\alpha(2T + 1)$
1	$< 1 >$	1(2)
3	$< 21 >$	1(4) 1(2)
5	$< 2^2 1 >$	1(6) 1(4) 1(2)
7	$< 2^3 1 >$	1(8) 1(6) 1(4) 1(2)
9	$< 2^4 1 >$	1(10) 1(8) 1(6) 1(4) 1(2)
11	$< 2^5 1 >$	1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
13	$< 2^6 1 >$	1(14) 1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
15	$< 2^7 1 >$	1(16) 1(14) 1(12) 1(10) 1(8) 1(6) 1(4) 1(2)

TABLE VI. $Sp(2\Omega)$ irreps $<\tilde{\mu}_S>$ that contain the $O(\Omega)$ irrep $[2]$, with multiplicity and $2T+1$ values listed in column 3.

		$[\tilde{\mu}] = [2]$
v_S	$<\tilde{\mu}_S>$	$\alpha(2T+1)$
2	$< 2 >$	1(3)
	$< 1^2 >$	1(1)
4	$< 2^2 >$	1(5) 1(3) 1(1)
	$< 21^2 >$	1(3)
6	$< 2^3 >$	1(7) 1(5) 2(3)
	$< 2^21^2 >$	1(5) 1(1)
8	$< 2^4 >$	1(9) 1(7) 2(5) 1(3) 1(1)
	$< 2^31^2 >$	1(7) 1(3)
10	$< 2^5 >$	1(11) 1(9) 2(7) 1(5) 2(3)
	$< 2^41^2 >$	1(9) 1(5) 1(1)
12	$< 2^6 >$	1(13) 1(11) 2(9) 1(7) 2(5) 1(3) 1(1)
	$< 2^51^2 >$	1(11) 1(7) 1(3)
14	$< 2^7 >$	1(15) 1(13) 2(11) 1(9) 2(7) 1(5) 2(3)
	$< 2^61^2 >$	1(13) 1(9) 1(5) 1(1)
16	$< 2^8 >$	1(17) 1(15) 2(13) 1(11) 2(9) 1(7) 2(5) 1(3) 1(1)
	$< 2^71^2 >$	1(15) 1(11) 1(7) 1(3)

TABLE VII. $Sp(2\Omega)$ irreps $<\tilde{\mu}_S>$ that contain the $O(\Omega)$ irrep $[1^2]$ with multiplicity and $2T+1$ values listed in columns 3.

		$[\tilde{\mu}] = [1^2]$
v_S	$<\tilde{\mu}_S>$	$\alpha(2T+1)$
2	$<2>$	1(1)
	$<1^2>$	1(3)
4	$<2^2>$	1(3)
	$<21^2>$	1(5) 1(3) 1(1)
6	$<2^3>$	1(5) 1(1)
	$<2^21^2>$	1(7) 1(5) 2(3)
8	$<2^4>$	1(7) 1(3)
	$<2^31^2>$	1(9) 1(7) 2(5) 1(3) 1(1)
10	$<2^5>$	1(9) 1(5) 1(1)
	$<2^41^2>$	1(11) 1(9) 2(7) 1(5) 2(3)
12	$<2^6>$	1(11) 1(7) 1(3)
	$<2^51^2>$	1(13) 1(11) 2(9) 1(7) 2(5) 1(3) 1(1)
14	$<2^7>$	1(13) 1(9) 1(5) 1(1)
	$<2^61^2>$	1(15) 1(13) 2(11) 1(9) 2(7) 1(5) 2(3)
16	$<2^8>$	1(15) 1(11) 1(7) 1(3)
	$<2^71^2>$	1(17) 1(15) 2(13) 1(11) 2(9) 1(7) 2(5) 1(3) 1(1)

TABLE VIII. $Sp(2\Omega)$ irreps $< \tilde{\mu}_S >$ that contain the $O(\Omega)$ irrep [3], with multiplicities and associated $(2T+1)$ values given in column 3.

v_S	$< \tilde{\mu}_S >$	$[\tilde{\mu}] = [3]$ $\alpha(2T+1)$
3	$< 21 >$	1(2)
5	$< 2^2 1 >$	1(4) 1(2)
7	$< 2^3 1 >$	1(6) 1(4) 1(2)
9	$< 2^4 1 >$	1(8) 1(6) 1(4) 1(2)
11	$< 2^5 1 >$	1(10) 1(8) 1(6) 1(4) 1(2)
13	$< 2^6 1 >$	1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
15	$< 2^7 1 >$	1(14) 1(12) 1(10) 1(8) 1(6) 1(4) 1(2)

TABLE IX. $Sp(2\Omega)$ irreps $< \tilde{\mu}_S >$ that contain the $O(\Omega)$ irrep [21], with multiplicities and associated $(2T+1)$ values given in column 3.

v_S	$< \tilde{\mu}_S >$	$[\tilde{\mu}] = [21]$
		$\alpha(2T+1)$
3	$< 21 >$	1(4) 1(2)
	$< 1^3 >$	1(2)
5	$< 2^2 1 >$	1(6) 2(4) 2(2)
	$< 21^3 >$	1(4) 1(2)
7	$< 2^3 1 >$	1(8) 2(6) 3(4) 2(2)
	$< 2^2 1^3 >$	1(6) 1(4) 1(2)
9	$< 2^4 1 >$	1(10) 2(8) 3(6) 3(4) 2(2)
	$< 2^3 1^3 >$	1(8) 1(6) 1(4) 1(2)
11	$< 2^5 1 >$	1(12) 2(10) 3(8) 3(6) 3(4) 2(2)
	$< 2^4 1^3 >$	1(10) 1(8) 1(6) 1(4) 1(2)
13	$< 2^6 1 >$	(14) 2(12) 3(10) 3(8) 3(6) 3(4) 2(2)
	$< 2^5 1^3 >$	1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
15	$< 2^7 1^1 >$	1(16) 2(14) 3(12) 3(10) 3(8) 3(6) 3(4) 2(2)
	$< 2^6 1^3 >$	1(14) 1(12) 1(10) 1(8) 1(6) 1(4) 1(2)

TABLE X. $Sp(2\Omega)$ irreps $< \tilde{\mu}_S >$ that contain the $O(\Omega)$ irrep $[1^3]$, with multiplicities and associated $(2T + 1)$ values given in column 3.

		$[\tilde{\mu}] = [1^3]$
v_S	$< \tilde{\mu}_S >$	$\alpha(2T+1)$
3	$< 21 >$	1(2)
	$< 1^3 >$	1(4)
5	$< 2^2 1 >$	1(4) 1(2)
	$< 21^3 >$	1(6) 1(4) 1(2)
7	$< 2^3 1 >$	1(6) 1(4) 1(2)
	$< 2^2 1^3 >$	1(8) 1(6) 2(4) 1(2)
9	$< 2^4 1 >$	1(8) 1(6) 1(4) 1(2)
	$< 2^3 1^3 >$	1(10) 1(8) 2(6) 2(4) 1(2)
11	$< 2^5 1 >$	1(10) 1(8) 1(6) 1(4) 1(2)
	$< 2^4 1^3 >$	1(12) 1(10) 2(8) 2(6) 2(4) 1(2)
13	$< 2^6 1 >$	1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
	$< 2^5 1^3 >$	1(14) 1(12) 2(10) 2(8) 2(6) 2(4) 1(2)
15	$< 2^7 1 >$	1(14) 1(12) 1(10) 1(8) 1(6) 1(4) 1(2)
	$< 2^6 1^3 >$	1(16) 1(14) 2(12) 2(10) 2(8) 2(6) 2(4) 1(2)

TABLE XI. Quantum numbers in the symmetry limit-II [defined by the group chain (20)] for $m = 12$ and $v = 2$. In the last two columns of the table, different T bands are separated by a semicolon. In the table v_S^r indicates that the multiplicity of the irrep (v_S, t_S) is r.

t_S	S	v_S	$[\tilde{\mu}] = [2]$ T	$[\tilde{\mu}] = [1^2]$ T
0	0	4	$1; 0, 2$	1
		8	$1, 2, 3; 0, 2, 4$	$1, 3$
		12	$1, 2, 3, 4, 5; 0, 2, 4, 6$	$1, 3, 5$
	1	2	1	0
		6	$1, 2; 1, 3$	$0, 2$
		10	$1, 2, 3, 4; 1, 3, 5$	$0, 2, 4$
	2	4	$1; 0, 2$	1
		8	$1, 2, 3; 0, 2, 4$	$1, 3$
	3	2	1	0
		6	$1, 2; 1, 3$	$0, 2$
	4	4	$1; 0, 2$	1
	5	2	1	0
1	0	2	0	1
		6	$0, 2$	$1, 2; 1, 3$
		10	$0, 2, 4$	$1, 2, 3, 4; 1, 3, 5$
	1	2	0	1
		4^2	1	$1; 0, 2$
		6	$0, 2$	$1, 2; 1, 3$
		8^2	$1, 3$	$1, 2, 3; 0, 2, 4$

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	10	$0, 2, 4$		$1, 2, 3, 4; 1, 3, 5$
	12	$1, 3, 5$		$1, 2, 3, 4, 5; 0, 2, 4, 6$
2	2^2	0		1
	4	1		$1; 0, 2$
	6^2	$0, 2$		$1, 2; 1, 3$
	8	$1, 3$		$1, 2, 3; 0, 2, 4$
	10	$0, 2, 4$		$1, 2, 3, 4; 1, 3, 5$
3	2	0		1
	4^2	1		$1; 0, 2$
	6	$0, 2$		$1, 2; 1, 3$
	8	$1, 3$		$1, 2, 3; 0, 2, 4$
4	2^2	0		1
	4	1		$1; 0, 2$
	6	$0, 2$		$1, 2; 1, 3$
5	2	0		1
	4	1		$1; 0, 2$
6	2	0		1

TABLE XII. Quantum numbers in the symmetry limit-III [defined by the group chain (29)] for $m = 12$ and $v = 2$. For other details see Table XI.

		$t_T = 0, \ [\tilde{\mu}] = [2]$	$t_T = 1, \ [\tilde{\mu}] = [1^2]$
S	v_T	T	T
0	4	$0, 2, 4$	$1, 2, 3, 4; 1, 3, 5$
	8	$0, 2$	$1, 2; 1, 3$
	12	0	1
1	2	$1, 3, 5$	$1, 2, 3, 4, 5; 0, 2, 4, 6$
	4	$0, 2, 4$	$1, 2, 3, 4; 1, 3, 5$
	6^2	$1, 3$	$1, 2, 3; 0, 2, 4$
	8	$0, 2$	$1, 2; 1, 3$
	10^2	1	$1; 0, 2$
	12	0	1
2	4	$0, 2, 4$	$1, 2, 3, 4; 1, 3, 5$
	6	$1, 3$	$1, 2, 3; 0, 2, 4$
	8^2	$0, 2$	$1, 2; 1, 3$
	10	1	$1; 0, 2$
	12^2	0	1
3	6	$1, 3$	$1, 2, 3; 0, 2, 4$
	8	$0, 2$	$1, 2; 1, 3$
	10^2	1	$1; 0, 2$
	12	0	1
4	8	$0, 2$	$1, 2; 1, 3$
	10	1	$1; 0, 2$
	12^2	0	1
5	10	1	$1; 0, 2$

		$t_T = 0, \ [\tilde{\mu}] = [1^2]$	$t_T = 1, \ [\tilde{\mu}] = [2]$
	12	0	1
6	12	0	1
0	2	$1, 3, 5$	$1, 2, 3, 4, 5; 0, 2, 4, 6$
	6	$1, 3$	$1, 2, 3; 0, 2, 4$
	10	1	$1; 0, 2$
1	4	$0, 2, 4$	$1, 2, 3, 4; 1, 3, 5$
	8	$0, 2$	$1, 2; 1, 3$
	12	0	1
2	6	$1, 3$	$1, 2, 3; 0, 2, 4$
	10	1	$1; 0, 2$
3	8	$0, 2$	$1, 2; 1, 3$
	12	0	1
4	10	1	$1; 0, 2$
5	12	0	1